

C^0 -rigidity of Poisson brackets

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Abstract

Consider a functional associating to a pair of compactly supported smooth functions on a symplectic manifold the maximum of their Poisson bracket. We show that this functional is lower semi-continuous with respect to the product uniform (C^0) norm on the space of pairs of such functions. This extends previous results of Cardin-Viterbo and Zapolsky. The proof involves theory of geodesics of the Hofer metric on the group of Hamiltonian diffeomorphisms. We also discuss a failure of a similar semi-continuity phenomenon for multiple Poisson brackets of three or more functions.

1 Statement of results

The subject of this note is function theory on symplectic manifolds. Let (M, ω) be a symplectic manifold (open or closed). Denote by $C_c^\infty(M)$ the space of smooth compactly supported functions on M and by $\|\cdot\|$ the standard *uniform norm* (also called the C^0 -norm) on it: $\|F\| := \max_{x \in M} |F(x)|$.

The definition of the Poisson bracket $\{F, G\}$ of two smooth functions $F, G \in C_c^\infty(M)$ involves first derivatives of the functions. Thus *a priori* there is no restriction on possible changes of $\{F, G\}$ when F and G are slightly perturbed in the uniform norm. Amazingly such restrictions do exist: this was first pointed out by F.Cardin and C.Viterbo [3] who showed that

$$\{F, G\} \not\equiv 0 \implies \liminf_{F', G' \xrightarrow{C^0} F, G} \|\{F', G'\}\| > 0.$$

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Our main result is as follows:

Theorem 1.1.

$$\max\{F, G\} = \liminf_{F', G' \xrightarrow{C^0} F, G} \max\{F', G'\} \quad (1)$$

for any symplectic manifold M and any pair $F, G \in C_c^\infty(M)$.

Replacing F by $-F$, we get a similar result for $-\min\{F, G\}$. In particular, this yields

$$\|\{F, G\}\| = \liminf_{F', G' \xrightarrow{C^0} F, G} \|\{F', G'\}\|, \quad (2)$$

which should be considered as a refinement of the Cardin-Viterbo theorem, and which gives a positive answer to a question posed in [5].

In the case $\dim M = 2$ formula (2) was first proved by F.Zapolsky [16] by methods of two-dimensional topology.

A generalization of the Cardin-Viterbo result in a different direction has been found by V.Humilière [7].

Remark 1.2. Note that (2) does *not* imply that $\{F', G'\} \xrightarrow{C^0} \{F, G\}$ when $F', G' \xrightarrow{C^0} F, G$ – see e.g. [7] for counterexamples.

Remark 1.3. Clearly \liminf cannot be replaced in the theorem by \lim : the maximum of the Poisson bracket of two functions can be arbitrarily increased by arbitrarily C^0 -small perturbations of the functions.

In the proof of Theorem 1.1 we use the following ingredient from “hard” symplectic topology: Denote by $Ham^c(M)$ the group of Hamiltonian diffeomorphisms of M generated by Hamiltonian flows with compact support. Then sufficiently small segments of one-parameter subgroups of the group $Ham^c(M)$ of Hamiltonian diffeomorphisms of M minimize the “positive part of the Hofer length” among all paths on the group in their homotopy class with fixed end points. This was proved by D.McDuff in [11, Proposition 1.5] for closed manifolds and in [12, Proposition 1.7] for open ones; see also papers [1], [9], [4], [10], [8] [13] for related results in this direction.

After an early draft of this paper has been written, L.Buhovsky found a different proof of Theorem 1.1 based on an ingenious application of the

energy-capacity inequality. Buhovsky's method enables him to give a quantitative estimate on the rate of convergence in the right-hand side of (1). These results will appear in a forthcoming article [2].

The next result gives an evidence for a failure of C^0 -rigidity for multiple Poisson brackets.

Theorem 1.4. *Let M be a symplectic manifold. There exists a constant $N \in \mathbb{N}$, depending only on the dimension of M , such that **for any** smooth functions $F_1, \dots, F_N \in C_c^\infty(M)$ there exist $F'_1, \dots, F'_N \in C_c^\infty(M)$ arbitrarily close in the uniform norm, respectively, to F_1, \dots, F_N which satisfy the following relation:*

$$\{F'_1, \{F'_2, \dots \{F'_{N-1}, F'_N\}\} \dots\} \equiv 0.$$

We shall see in Section 2.3 below that in the case $\dim M = 2$ the result above holds for $N = 3$.

Question 1.5. Does the theorem above remain valid with $N = 3$ on an arbitrary symplectic manifold?

The following claim, though it does not answer Question 1.5, shows that Theorem 1.1 cannot be formally extended to the triple Poisson bracket.

Theorem 1.6. *For any symplectic manifold M **one can find** 3 functions $F, G, H \in C_c^\infty(M)$ satisfying $\{F, \{G, H\}\} \neq 0$ such that there exist smooth functions $F', G', H' \in C_c^\infty(M)$ arbitrarily close in the uniform norm, respectively, to F, G, H and satisfying the condition*

$$\{F', \{G', H'\}\} \equiv 0.$$

The theorem will be proved in Section 2.3. The proof shows that the phenomenon is local: we just implant a 2-dimensional example (see the remark after Theorem 1.4) in a Darboux chart.

Surprisingly, the next problem is open even in dimension 2:

Problem 1.7. Compare

$$\liminf_{F', G' \xrightarrow{C^0} F, G} \max\{\{F', G'\}, G'\}$$

with $\max\{\{F, G\}, G\}$ for some/all pairs of functions F, G on some/all symplectic manifolds.

2 Proofs

2.1 Preliminaries

Given a (time-dependent) Hamiltonian $H : M \times [0, 1] \rightarrow \mathbb{R}$, denote by X_H the (time-dependent) Hamiltonian vector field generated by H . The Poisson bracket of two functions $F, G \in C_c^\infty(M)$ is defined by $\{F, G\} = dF(X_G)$.

Let $\text{Ham}^c(M)$ be the group of Hamiltonian diffeomorphisms of (M, ω) generated by compactly supported (time-dependent) Hamiltonians. Write $\widetilde{\text{Ham}}^c(M)$ for the universal cover of $\text{Ham}^c(M)$, where the base point is chosen to be the identity map $\mathbb{1}$. Denote by ψ_H^t , $t \in \mathbb{R}$, the Hamiltonian flow generated by H (i.e. the flow of X_H). Let $\psi_H := \psi_H^1$ and let $\tilde{\psi}_H \in \widetilde{\text{Ham}}^c(M)$ be the lift of ψ_H associated to the path $\{\psi_H^t\}, t \in [0; 1]$. We will say that ψ_H and $\tilde{\psi}_H$ are generated by H . We will also denote $\|H\| = \max_{M \times [0, 1]} |H(x, t)|$ (for time-independent Hamiltonians this norm coincides with the uniform norm on $C_c^\infty(M)$ introduced above). Set $H_t = H(\cdot, t)$.

Recall that the flow $\psi_H^t \psi_K^t$ is generated by the Hamiltonian $H \sharp K(x, t) = H(x, t) + K((\psi_H^t)^{-1}x, t)$ and the flow $\psi_H \psi_K^t (\psi_H)^{-1}$ by $K((\psi_H)^{-1}x, t)$.

A Hamiltonian H on $M \times [0; 1]$ is called *normalized* if either M is open and $\bigcup_t \text{support}(H_t)$ is contained in a compact subset of M , or M is closed and H_t has zero mean for all t . The set of all normalized Hamiltonian functions is denoted by \mathcal{F} . Note that if $H, K \in \mathcal{F}$ then both $H \sharp K$ and $K((\psi_H)^{-1}x, t)$ also belong to \mathcal{F} .

For $a, b \in \widetilde{\text{Ham}}^c(M, \omega)$ write $[a, b]$ for the commutator $aba^{-1}b^{-1}$.

Lemma 2.1. *Assume $H, K \in C_c^\infty(M)$ are time-independent Hamiltonians. Then $[\tilde{\psi}_H, \tilde{\psi}_K]$ can be generated by $L(x, t) = H(x) - H(\psi_K^{-1}\psi_H^{-t}x)$.*

Proof. It is easy to see that the element $[\tilde{\psi}_H, \tilde{\psi}_K] \in \widetilde{\text{Ham}}^c(M)$ can be represented by the path $\{\psi_H^t \psi_K \psi_H^{-t} \psi_K^{-1}\}$ where $t \in [0; 1]$. The flow ψ_H^{-t} is generated by $-H$ and therefore the flow $\psi_K \psi_H^{-t} \psi_K^{-1}$ is generated by the Hamiltonian $-H \circ \psi_K^{-1}$. Thus the flow $\psi_H^t \psi_K \psi_H^{-t} \psi_K^{-1}$ is generated by $H \sharp (-H \circ \psi_K^{-1})(x, t) = H(x) - H(\psi_K^{-1} \psi_H^{-t} x)$. \square

The group $\widetilde{\text{Ham}}^c(M)$ carries conjugation-invariant functionals ρ^+ and ρ defined by

$$\rho^+(\tilde{\psi}) := \inf_H \int_0^1 \max_{x \in M} H(x, t) dt$$

and

$$\rho(\tilde{\psi}) := \inf_H \int_0^1 (\max_{x \in M} H(x, t) - \min_{x \in M} H(x, t)) dt ,$$

where the infimum is taken over all Hamiltonians $H \in \mathcal{F}$ generating $\tilde{\psi}$. The functional ρ is the Hofer (semi)-norm [6] (see e.g. [14] for an introduction to Hofer's geometry). It gives rise to the bi-invariant Hofer (pseudo-)metric on $\widetilde{Ham^c(M)}$ by $d(\tilde{\phi}, \tilde{\psi}) = \rho(\tilde{\phi}^{-1}\tilde{\psi})$. The functional ρ^+ , which is sometimes called the “positive part of the Hofer norm”, satisfies the triangle inequality but is not symmetric. Note also that $\rho^+ \leq \rho$. We shall use the following properties of these functionals. By the triangle inequality for ρ^+

$$|\rho^+(\tilde{\phi}) - \rho^+(\tilde{\psi})| \leq \max(\rho^+(\tilde{\phi}^{-1}\tilde{\psi}), \rho^+(\tilde{\psi}^{-1}\tilde{\phi})) \leq d(\tilde{\phi}, \tilde{\psi}) . \quad (3)$$

This readily yields

$$|\rho^+(\tilde{\psi}_H) - \rho^+(\tilde{\psi}_K)| \leq d(\tilde{\psi}_H, \tilde{\psi}_K) \leq 2\|H - K\| \quad (4)$$

for any $H, K \in \mathcal{F}$. McDuff showed [11, Proposition 1.5], [12, Proposition 1.7] that for every *time-independent* function $H \in \mathcal{F}$ there exists $\delta > 0$ so that

$$\rho^+(\tilde{\psi}_{tH}) = t \cdot \max H \quad \forall t \in (0; \delta) . \quad (5)$$

Lemma 2.2. *Assume $H, K \in C_c^\infty(M)$ are time-independent Hamiltonians with zero mean. Then $\rho^+([\tilde{\psi}_H, \tilde{\psi}_K]) \leq \max\{H, K\}$.*

Proof. By Lemma 2.1 $[\tilde{\psi}_H, \tilde{\psi}_K]$ can be generated by

$$L(x, t) = H(x) - H(\psi_K^{-1}\psi_H^{-t}x).$$

Note that

$$\begin{aligned} \int_0^1 \max L(x, t) dt &= \int_0^1 \max(H - H \circ \psi_K^{-1} \circ \psi_H^{-t}) dt \\ &= \int_0^1 \max(H \circ \psi_H^t - H \circ \psi_K^{-1}) dt \\ &= \int_0^1 \max(H - H \circ \psi_K^{-1}) dt = \int_0^1 \max(H \circ \psi_K - H) dt \end{aligned}$$

since H is constant on the orbits of the flow ψ_H^t . Taking into account that

$$H(\psi_K x) - H(x) = \int_0^1 \frac{d}{dt} H(\psi_K^t x) dt = \int_0^1 \{H, K\}(\psi_K^t x) dt,$$

we get that

$$\rho^+([\tilde{\psi}_H, \tilde{\psi}_K]) \leq \int_0^1 \max L(x, t) dt \leq \max\{H, K\},$$

which yields the lemma. \square

2.2 Proof of Theorem 1.1

We assume without loss of generality that all the functions F_i, G_i, F, G are normalized.

Denote by $f_s, g_t, s, t \in [0, 1]$, the Hamiltonian flows generated by F and G , and by \tilde{f}_s, \tilde{g}_t their respective lifts to $\widetilde{Ham^c(M)}$. Note that for fixed s and t the elements \tilde{f}_s and \tilde{g}_t are generated, respectively, by the Hamiltonians sF and tG .

By Lemma 2.1 for fixed s, t the commutator $[\tilde{f}_s, \tilde{g}_t]$ can be generated by the Hamiltonian $L_{s,t}(x, \tau) = sF(x) - sF(g_t^{-1} f_{\tau s}^{-1} x)$ (use Lemma 2.1 with $H = sF, K = tG$ and note that $\psi_{sF}^\tau = f_{\tau s}$). Clearly $L_{s,t} \in \mathcal{F}$ since $F, G \in \mathcal{F}$.

Lemma 2.3. $L_{s,t} = st\{F, G\} + K_{s,t}$, where $\|K_{s,t}\|/st \rightarrow 0$ as $s, t \rightarrow 0$.

Proof. We need to compute the relevant terms in the expansion of $L_{s,t}$ with respect to s, t at $s = 0, t = 0$.

Clearly, $L_{0,0} \equiv 0$.

The first order terms are as follows:

$$\begin{aligned} \frac{\partial L_{s,t}}{\partial s}(x, \tau) &= \partial(sF(x) - sF(g_t^{-1} f_{\tau s}^{-1} x))/\partial s = \\ &= F(x) - F(g_t^{-1} f_{\tau s}^{-1} x) - sdF \circ dg_t^{-1}(X_{-sF}(x)) = \\ &= F(x) - F(g_t^{-1} f_{\tau s}^{-1} x) + s^2 dF \circ dg_t^{-1}(X_F(x)), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial L_{s,t}}{\partial t}(x, \tau) &= \partial(sF(x) - sF(g_t^{-1} f_{\tau s}^{-1} x))/\partial t = \\ &= -sdF(X_{-G}(f_{\tau s}^{-1} x)) = s\{F, G\}(f_{\tau s}^{-1} x). \end{aligned}$$

Evaluating $\partial L_{s,t}/\partial s(x, \tau)$ and $\partial L_{s,t}/\partial t(x, \tau)$, respectively, at the points $(s, 0)$ and $(0, t)$ (for a fixed (x, τ)) we see that

$$\frac{\partial L_{s,0}}{\partial s}(x, \tau) \equiv 0$$

(since F is constant on the orbits of the flow f_s , $s \in \mathbb{R}$) and

$$\frac{\partial L_{0,t}}{\partial t}(x, \tau) \equiv 0.$$

Thus

$$\frac{\partial^k}{\partial s^k} \Big|_{(s,t)=(0,0)} L_{s,t}(x, \tau) = 0 = \frac{\partial^k}{\partial t^k} \Big|_{(s,t)=(0,0)} L_{s,t}(x, \tau), \text{ for any } k \geq 1.$$

Finally, let us compute $\frac{\partial^2}{\partial s \partial t} \Big|_{(s,t)=(0,0)} L_{s,t}(x, \tau)$:

$$\begin{aligned} \frac{\partial^2}{\partial s \partial t} \Big|_{(s,t)=(0,0)} L_{s,t}(x, \tau) &= \frac{\partial}{\partial s} \Big|_{s=0} \frac{\partial L_{s,0}}{\partial t}(x, \tau) = \\ &= \frac{\partial}{\partial s} \Big|_{s=0} s \{F, G\}(f_{\tau s}^{-1} x) = \{F, G\}(x). \end{aligned}$$

This finishes the proof of the lemma. \square

Now we are ready to complete the proof of Theorem 1.1. The inequality

$$\max\{F, G\} \geq \liminf_{F', G' \xrightarrow{C^0} F, G} \max\{F', G'\}$$

is trivial so we only need to prove the opposite one. Let F_i, G_i be sequences of smooth functions such that

$$F_i, G_i \xrightarrow{C^0} F, G, \quad i \rightarrow +\infty,$$

and

$$\max\{F_i, G_i\} \rightarrow A, \quad i \rightarrow +\infty.$$

We need to show that $\max\{F, G\} \leq A$.

Assume on the contrary that $\max\{F, G\} > A$. Pick B such that $A < B < \max\{F, G\}$. Then for any sufficiently large i

$$\max\{F_i, G_i\} \leq B.$$

Denote by $f_{s,i}, g_{t,i}$, respectively, the time- s and time- t maps of the flows generated by F_i and G_i . Their lifts to $\widetilde{Ham}^c(M)$ will be decorated by tildes. The right inequality in (4) easily implies that the sequences $\widetilde{f}_{s,i}$ and $\widetilde{g}_{t,i}$ converge, respectively, to \widetilde{f}_s and \widetilde{g}_s in the Hofer (pseudo-)metric. Since by (3) the functional ρ^+ is continuous in the Hofer (pseudo-)metric,

$$\rho^+([\widetilde{f}_{s,i}, \widetilde{g}_{t,i}]) \rightarrow \rho^+([\widetilde{f}_s, \widetilde{g}_s]) \text{ as } i \rightarrow \infty.$$

By Lemma 2.2,

$$\rho^+([\widetilde{f}_{s,i}, \widetilde{g}_{t,i}]) \leq st \cdot \max\{F_i, G_i\} \leq stB$$

for any sufficiently large i . Hence, taking the limit in the left-hand side as $i \rightarrow +\infty$, we get

$$\rho^+([\widetilde{f}_s, \widetilde{g}_s]) \leq stB. \quad (6)$$

Choose $\epsilon > 0$ such that $B + 2\epsilon < \max\{F, G\}$. Take sufficiently small $s, t > 0$ so that the function $K_{s,t}$ from Lemma 2.3 admits a bound

$$\|K_{s,t}\| \leq \epsilon st \quad (7)$$

and so that the Hamiltonian $st\{F, G\}$ is sufficiently small and satisfies

$$\rho^+(\widetilde{\psi}_{st\{F,G\}}) = st \cdot \max\{F, G\}, \quad (8)$$

see formula (5). Lemma 2.3 and inequalities (7), (4) yield

$$|\rho^+([\widetilde{f}_s, \widetilde{g}_s]) - \rho^+(\widetilde{\psi}_{st\{F,G\}})| \leq 2\epsilon st.$$

Hence,

$$\rho^+([\widetilde{f}_s, \widetilde{g}_s]) \geq \rho^+(\widetilde{\psi}_{st\{F,G\}}) - 2\epsilon st = st(\max\{F, G\} - 2\epsilon).$$

Combining this with (6), we get

$$st(\max\{F, G\} - 2\epsilon) \leq \rho^+([\widetilde{f}_s, \widetilde{g}_s]) \leq stB,$$

and hence

$$\max\{F, G\} - 2\epsilon \leq B$$

which contradicts our choice of B and ϵ . We have obtained a contradiction. Hence $\max\{F, G\} \leq A$ and the theorem is proven. \square

2.3 Proofs of Theorems 1.4, 1.6

Proof of Theorem 1.4.

For simplicity we will prove the result in the case $\dim M = \mathbb{T}^2$ with $N = 3$. The general case can be done in a similar way using [15].

Define a *thick grid* T with mesh c in M as a union of pair-wise disjoint squares on M such that each square has a side $2c$ and the centers of the squares form a rectangular grid with the mesh $3c$. A *T-tamed function* is a smooth function which is constant in a small neighborhood of each square of the thick grid T (but its values may vary from square to square).

One can easily construct a sequence $c_i \rightarrow 0$ and $N = 3$ thick grids U_i, V_i, W_i with mesh c_i so that $U_i \cup V_i \cup W_i = M$ for all i . (See [15] on how to construct a similar covering of an arbitrary M by a number of thick grids depending only on $\dim M$).

Now for every $\epsilon > 0$ there exists i large enough so that every triple of functions $F_1, F_2, F_3 \in C_c^\infty(M)$ can be ϵ -approximated, respectively, by U_i, V_i, W_i -tamed functions $F'_1, F'_2, F'_3 \in C_c^\infty(M)$. Take any point $x \in M$. Then at least one of the functions F'_1, F'_2, F'_3 is constant near x . Thus $\{F'_1, \{F'_2, F'_3\}\} \equiv 0$, and the claim follows. \square

Proof of Theorem 1.6.

Assume $\dim M = 2n > 2$ (the case $\dim M = 2$ has been dealt with in the proof of Theorem 1.4). In a local Darboux chart with coordinates $p_1, q_1, \dots, p_n, q_n$ on M choose an open cube

$$P = K^{2n-2} \times K^2,$$

where K^{2n-2} is an open cube in the $(p_1, q_1, \dots, p_{n-1}, q_{n-1})$ -coordinate plane and K^2 is a open square in the (p_n, q_n) -coordinate plane. Fix a smooth compactly supported non-zero function χ on K^{2n-2} . Given a smooth compactly supported function L on K^2 , define the function $\chi L \in C_c^\infty(M)$ as

$$\chi L(p_1, q_1, \dots, p_n, q_n) := \chi(p_1, q_1, \dots, p_{n-1}, q_{n-1}) L(p_n, q_n)$$

on P and as zero outside P .

Now pick any functions $F_1, G_1, H_1 \in C_c^\infty(K^2)$ such that

$$\{F_1, \{G_1, H_1\}\} \neq 0.$$

Set

$$F := \chi F_1, G := \chi G_1, H := \chi H_1 \in C_c^\infty(M).$$

As in the proof of Theorem 1.4 (note that in the case of the two-dimensional square the construction of the thick grids is as easy as in the case of \mathbb{T}^2), choose C^0 -small perturbations $F'_1, G'_1, H'_1 \in C_c^\infty(K^2)$ of F_1, G_1, H_1 so that

$$\{F'_1, \{G'_1, H'_1\}\} \equiv 0.$$

Then $F' := \chi F'_1, G' := \chi G'_1, H' := \chi H'_1 \in C_c^\infty(M)$ satisfy

$$\{F', \{G', H'\}\} = \{\chi F'_1, \{\chi G'_1, \chi H'_1\}\} = \chi^3 \{F'_1, \{G'_1, H'_1\}\} \equiv 0,$$

because of the Leibniz rule for Poisson brackets and because the Poisson bracket of χ and any function of p_n, q_n vanishes identically. For the same reason

$$\{F, \{G, H\}\} = \{\chi F_1, \{\chi G_1, \chi H_1\}\} = \chi^3 \{F_1, \{G_1, H_1\}\} \neq 0.$$

Clearly, by choosing F'_1, G'_1, H'_1 arbitrarily C^0 -close to F_1, G_1, H_1 in $C_c^\infty(K^2)$ we can turn F', G', H' into arbitrarily C^0 -small perturbations of F, G, H in $C_c^\infty(M)$. Thus we have constructed F, G, H, F', G', H' satisfying the required conditions. \square

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